Series Solutions

Consider $\quad a_2(x)\dfrac{d^2y}{dx^2} + a_1(x)\dfrac{dy}{dx} + a_0(x)\,y = 0$

and suppose this DE has a solution which can be written as an infinite series,

$$\sum_{n=0}^{\infty} C_n(x-x_0)^n = C_0 + C_1(x-x_0) + C_2(x-x_0)^2 + \cdots + C_n(x-x_0)^n + \cdots$$

where C's are constants. Often this is referred to as a power series (PS) "in" $x-x_0$ or a PS "about" x_0 or a PS centered at x_0.

We will need to in general find the coefficients (the C's) and also address the question of if we have a solution at all.

We can rewrite as $\quad \dfrac{d^2y}{dx^2} + P_1(x)\dfrac{dy}{dx} + P_2(x)\,y = 0$

where $P_1(x) = \dfrac{a_1(x)}{a_2(x)}$ & $P_2(x) = \dfrac{a_0(x)}{a_2(x)}$

<u>Def</u>: A function f is <u>analytic</u> @ x_0 if its Taylor Series centered at x_0

$$\sum_{n=0}^{\infty} \frac{f^{n}(x_0)}{n!}(x-x_0)^n$$

exists and converges to $f(x)$ $\forall x$ in some open interval including x_0.

It is noted that polynomials, exponentials (ex e^x) $\sin x$, $\cos x$ are analytic everywhere (for all x).

A rational function is analytic except for those x values which make the denominator zero.

ex.) $\dfrac{1}{x^2-5}$ is analytic except at $x = \pm\sqrt{5}$.

<u>Def</u>: x_0 is an <u>ordinary point</u> of a DE if <u>both</u> $P_1(x)$ & $P_2(x)$ are analytic at x_0. If either or both $P_1(x)$ and or $P_2(x)$ is not analytic at x_0, then we call x_0 a <u>singular</u> point.

ex.) $$\frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2+2)y = 0$$

$P_1(x) = x$ $\quad$ $P_2(x) = x^2+2$ both are polynomials and are analytic $\forall x$.

ex.) $$(x-5)\frac{d^2y}{dx^2} + x\frac{dy}{dx} + \frac{1}{x}y = 0$$

rewrite

$$\frac{d^2y}{dx^2} + \frac{x}{x-5}\frac{dy}{dx} + \frac{1}{x(x-5)}y = 0$$

now here $P_1(x) = \dfrac{x}{x-5}$ & $P_2(x) = \dfrac{1}{x(x-5)}$

$P_1(x)$ is analytic except at $x=5$ and $P_2(x)$ is analytic except at $x=0$, $x=5$.

So $x=0$ & $x=5$ are <u>singular points</u>.

All other values of x are <u>ordinary points</u>.

Notice $P_1(x)$ is analytic at $x=0$ but we say $x=0$ is a singular point for the DE. <u>Both</u> $P_1(x)$ & $P_2(x)$ must be analytic @ x_0 for x_0 to be an ordinary point.

We are now ready for this statement.

If x_0 is an ordinary point for our 2^{nd} order DE, then the DE has two PS solutions and these two solns are linearly Independent! The general solution is a linear combination of these two linearly indep. PS.

So in the first example

$$\frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2+2)y = 0 \quad \text{all values}$$

of x are ordinary points so this DE has two linearly indep. solns about ANY point x_0.

in the 2^{nd} ex.) $\dfrac{d^2y}{dx^2} + \dfrac{x}{x-5}\dfrac{dy}{dx} + \dfrac{1}{x(x-5)}y = 0$

$x = 0$ & 5 are singular points so for any $x \neq 0$ or 5 we will have two linearly indep. solns of this DE.

We know we will have a soln. of the form $\sum_{n=0}^{\infty} C_n(x-2)^n$ about ordinary pt $x=2$.

We do not know if there is a soln. of the form $\sum_{n=0}^{\infty} C_n x^n$ or $\sum_{n=0}^{\infty} C_n(x-5)^n$ about the singular points $x=0, 5$. We may, but no guarantees.

So now how do we find a solution?